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Two categories of effective continuous cpos

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Abstract

This paper presents two categories of effective continuous complete partial orders (cpos). We define a new criterion on the basis of a cpo so as to make the resulting category of consistently complete continuous cpos cartesian closed. We also generalise to continuous cpos the definition of a complete set, which was used as a definition of effective bifinite domains in Hamrin and Stoltenberg-Hansen [Cartesian closed categories of effective domains, in: H. Schwichtenberg, R. Steinbrüggen (Eds.), Proof and System-Reliability, Kluwer Academic Publishers, Dordrecht, 2002, pp. 1–20], and investigate the closure results that can be obtained.

MSC: 03D45

Keywords: Domain theory; Cartesian closure; Effective domains

1. Introduction

This paper develops a theory of *effective continuous complete partial orders* (cpos). The theory of cpos or *domain theory* started with the works of Scott (see for example [18,19]) and Ershov (see [6,8]).

An important motivation for the interest in effective domains is the use of domain theory to study computablility on continuous structures, i.e. structures used in analysis (see for example [2,22,23,4]).

The theory of effective domains has been thoroughly developed for consistently complete algebraic cpos, beginning in for example [5,17,16]. It is relatively easy to construct cartesian closed categories of effective consistently complete algebraic cpos.

Although effective consistently complete algebraic cpos suffice for many applications they do not suffice for all. For example, Blanck [3] used the Plotkin power domain construction in order to deal with the space of compact subsets of a metric space. It is well known that the Plotkin power domain construction takes you out of the consistently complete algebraic cpos, but stays within the bifinite domains. Furthermore, much of this type of work uses continuous cpos, a class properly containing the algebraic cpos.

In this paper we consider the case when the cpo is merely continuous. It is natural to retain an analogous notion of effectivity as for effective consistently complete algebraic cpos. The category of consistently complete continuous cpos is cartesian closed. However, it seems necessary to impose new restrictions on the basis in order to obtain a cartesian closed category of *effective* consistently complete cpos.

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We define two categories of continuous cpos and study the effective closure properties they have. We first discuss a new restriction on a basis which we call *almost algebraic*. We show that there is a cartesian closed category of effective consistently complete continuous cpos with closed and almost algebraic basis, the morphisms being the effective continuous functions. An inspiration for the notion of an almost algebraic basis is [20]. In that paper a basis with the *inverse approximation property* is considered, resulting in a cartesian closed subcategory of the consistently complete continuous cpos. The notion of an almost algebraic basis is also related to Tang [24], who considers conditions on a basis in order to obtain a cartesian closed category of continuous lattices. These extra conditions on a basis are under suitable assumptions equivalent to a basis being almost algebraic.

In our second attempt we combine almost algebraicity with the notion of a *c-bifinite* basis, thus obtaining the new notion of a *c-bifinite domain*. This is a generalisation of the characterisation of a bifinite domain in [11]. We show that the class of c-bifinite domains with almost algebraic basis is closed under the Plotkin power domain construction. Furthermore, we show that the function space of almost algebraic c-bifinite domains is c-bifinite.

The paper is organised as follows. We start by reviewing basic domain theory, category theory, numberings and effectivity for continuous cpos in Section 2. In Section 3 we present the definition of an almost algebraic basis, and obtain as a result a cartesian closed category of effective consistently complete continuous cpos. In Section 4 we define the c-bifinite domains and study the closure results that can be obtained. Finally, in Section 5 we define effectivity for c-bifinite domains, and prove effective versions of the theorems in Section 4.

2. Preliminaries

In this section we review basic domain theory, category theory and effective domain theory.

2.1. Domain theory

This section presents a background to domain theory. The reader can find more details and proofs in [1] and the textbook [21].

Let D be a set and let $\sqsubseteq$ be a binary relation on D. Then $(D, \sqsubseteq)$ is a *preorder* if $\sqsubseteq$ is reflexive and transitive. $(D, \sqsubseteq)$ is a *partial order* if $\sqsubseteq$ is in addition antisymmetric.

Let A and B be sets. As usual, we let $A \subseteq B$ denote that A is a subset of B. We also let $A \subseteq^* B$ ($A \subseteq_f B$) denote that A is a non-empty (finite) subset of B. Analogously, we let $\wp_f^*(B)$ denote the set of non-empty finite subsets of B.

A subset $A \subseteq D$ is *consistent in* D (with respect to $\sqsubseteq$) if

$$(\exists x \in D)(\forall a \in A)(a \sqsubseteq x).$$

That two elements x and y are consistent in D is sometimes denoted by $\mathrm{Cons}(x, y)$.

Let $D = (D; \sqsubseteq, \bot)$ be a partial order with a least element $\bot$. A set $A \subseteq D$ is *directed* if $A \neq \emptyset$ and $(\forall x, y \in A)(\exists z \in A)(x \sqsubseteq z \wedge y \sqsubseteq z)$. We let $\bigsqcup A$ denote the least upper bound (or supremum) of A if it exists. If $A = \{x, y\}$ then we write $x \sqcup y$ for the supremum of x and y whenever it exists. We say that D is a *cpo* if any directed set in D has a supremum in D. Let $(D, \sqsubseteq_D)$ and $(E, \sqsubseteq_E)$ be partial orders. Then the *cartesian product* $(D \times E, \sqsubseteq)$ of D and E is the partially ordered set

$$D \times E := \{(x, y) : x \in D \wedge y \in E\},$$

ordered by $(x, y) \sqsubseteq (z, w) :\Leftrightarrow x \sqsubseteq_D z \wedge y \sqsubseteq_E w$. It is a cpo when D and E are cpos.

Definition 1. Let $D = (D; \sqsubseteq, \bot)$ be a cpo. For $x, y \in D$ we say that x is way-below y (written $x \ll y$) if for each directed $A \subseteq D$,

$$y \sqsubseteq \bigsqcup A \Rightarrow (\exists z \in A)(x \sqsubseteq z).$$

An element x is compact if $x \ll x$. We let D_{c} denote the set of compact elements of D.

Lemma 2. *Let $D = (D; \sqsubseteq, \bot)$ be a cpo and let $x, y, z, w \in D$. Then*

(1) $x \ll y \Rightarrow x \sqsubseteq y$.

(2) $z \sqsubseteq x \ll y \sqsubseteq w \Rightarrow z \ll w$.

We let $\Downarrow A := \{x \in D : (\exists y \in A)(x \ll y)\}$ and write $\Downarrow x$ for $\Downarrow\{x\}$. Correspondingly, we let $\Uparrow A := \{x \in D : (\exists y \in A)(x \gg y)\}$ and write $\Uparrow x$ for $\Uparrow\{x\}$.

Definition 3. Let $D = (D; \sqsubseteq, \bot)$ be a cpo. A subset $B \subseteq D$ is a base (or basis) for D if for each $x \in D$,

$$\mathrm{approx}_B(x) := \{y \in B : y \ll x\}$$

is directed and $\bigsqcup \mathrm{approx}_B(x) = x$.

A basis B is *reduced* if for all $b \in B$ we have $\Uparrow b \neq \emptyset$. If B is a basis for a cpo D then $\{b \in B : \Uparrow b \neq \emptyset\}$ is also a basis for D. A basis B is *closed* if for all $b, c \in B$ such that $\mathrm{Cons}(b, c)$ we have that $b \sqcup c$ exists and $b \sqcup c \in B$.

Definition 4. A cpo $D = (D; \sqsubseteq, \bot)$ is continuous if it has a basis.

Note that $\ll$ on a continuous cpo D with a base B can be used to describe $\sqsubseteq$ on D, since $x \sqsubseteq y \Leftrightarrow \mathrm{approx}_B(x) \subseteq \mathrm{approx}_B(y)$.

The way-below relation satisfies the following *interpolation property* for continuous cpos, upon which many observations rely. Let $M \ll y$ denote that $(\forall z \in M)(z \ll y)$.

Lemma 5. *Let $D = (D; \sqsubseteq, \bot)$ be a continuous cpo with base B. Let $M \subseteq_f D$ and suppose $M \ll y$. Then there exists $x \in B$ such that $M \ll x \ll y$.*

The way-below relation is transitive and anti-symmetric. It is normally not reflexive. If it is reflexive on a base B for a cpo D then we have a minimal base, that is $B = D_c$.

Definition 6. A cpo $D = (D; \sqsubseteq, \bot)$ is algebraic if D_c is a base for D.

An *ideal* of a preorder $(P; \sqsubseteq)$ is a directed subset $I \subseteq P$, which is closed downwards. *The ideal completion* $\mathrm{Idl}(P)$ of the preorder P is the set of all ideals in P ordered by subset inclusion. It is an algebraic cpo.

2.2. The function space

In this section we describe the continuous functions between cpos.

Definition 7. Let D and E be cpos. $f : D \to E$ is a (Scott-)continuous function if f is monotone and for all directed sets $A \subseteq D$ we have $f(\bigsqcup A) = \bigsqcup f[A]$.

We denote the *function space*, the set of all continuous functions from D to E with the pointwise ordering, by $[D \to E]$. The function space is a cpo, in which for every directed set $\mathcal{F} \subseteq [D \to E]$ we have that $(\bigsqcup \mathcal{F})(x) = \bigsqcup\{f(x) : f \in \mathcal{F}\}$. We also have the following useful lemma for continuous functions between continuous cpos.

Lemma 8. *Let D and E be continuous cpos. Then $f : D \to E$ is continuous if and only if for all $x \in D$ and for all $b \ll f(x)$ there is $a \ll x$ such that $b \ll f(a)$.*

The simplest continuous functions are the (single) step functions $\langle a; b\rangle$.

Definition 9. The step function $\langle a; b\rangle : D \to E$, for $a \in D$ and $b \in E$, is defined by

$$\langle a; b\rangle(x) = \begin{cases} b & \text{if } a \ll x, \\ \bot & \text{otherwise.} \end{cases}$$

It is straightforward to see that $\langle a; b\rangle$ is continuous. We have the following relationship between step functions and continuous functions.

Proposition 10. *Let D and E be cpos and let $a \in D$ and $b \in E$.*
(1) *Suppose $f \in [D \to E]$. Then*

$$b \ll f(a) \Longrightarrow \langle a; b\rangle \ll f.$$

(2) *If D and E are continuous cpos with bases B_D and B_E and $f \in [D \to E]$ then*

$$f = \bigsqcup\{\langle a; b\rangle : a \in B_D,\ b \in B_E,\ b \ll f(a)\}.$$

Proof. For (1), suppose $f \sqsubseteq \bigsqcup \mathcal{F}$ for a directed $\mathcal{F} \subseteq [D \to E]$. Then we have $f(a) \sqsubseteq \bigsqcup\{g(a) : g \in \mathcal{F}\}$. By assumption, $b \sqsubseteq g(a)$ for some $g \in \mathcal{F}$ and hence $\langle a; b\rangle \sqsubseteq g$. For (2), suppose $\{\langle a; b\rangle : a \in B_D,\ b \in B_E,\ b \ll f(a)\}$ has g as an upper bound and let $x \in D$. Suppose $b \ll f(x)$. Then $b \ll f(a)$ for some $a \ll x$ by continuity. Hence $\langle a; b\rangle \sqsubseteq g$, and $\langle a; b\rangle(x) = b \sqsubseteq g(x)$. But $f(x) = \bigsqcup\{b : b \ll f(x)\} \sqsubseteq g(x)$, so $f \sqsubseteq g$. □

2.3. Cartesian closure for cpos

In this section we review some category theory for domains. The standard textbook reference for category theory is [14]. Recall that a category **C** is said to be *cartesian closed* if it has a terminal object and is closed under finite products and exponentiation. Let **CPO** be the category of cpos with continuous functions as morphisms. Then **CPO** is cartesian closed.

Due to the following lemma, whose complete proof appears in [13], it is easy to prove cartesian closure (when it holds) for the subcategories of **CPO** we are interested in.

Lemma 11. *Let* **C** *be a cartesian closed full subcategory of* **CPO** *and let D and E be objects of* **C**. *Then the following hold*:
(1) *The categorical product in* **C** *of D and E is isomorphic to the cartesian product $D \times E$.*
(2) *The exponential in* **C** *of D and E is isomorphic to $[D \to E]$.*

Note that we require that all cpos have a bottom element. Hence a terminal object does always exist. Most of the time it is trivial to construct the cartesian product for domains. Therefore, the previous lemma in practice reduces the problem of showing cartesian closure for **C** to the problem of showing that $[D \to E]$ is in **C** whenever $D, E \in$ **C**.

Neither **ALG**, the category of algebraic cpos with continuous functions as morphisms, nor **CONT**, the category of continuous cpos with continuous functions as morphisms, are cartesian closed. This means that we must restrict ourselves to smaller classes of domains. Historically this was first done for the consistently (or bounded) complete domains. As the name indicates, they have suprema for all bounded (and not necessarily only directed) subsets.

Definition 12. A cpo $D = (D; \sqsubseteq, \bot)$ is consistently complete if $x \sqcup y$ exists in D for $x, y \in D$ such that x and y are consistent.

Let **CDOM** be the category of consistently complete continuous cpos with continuous functions as morphisms.

We note the following proposition, important when considering the effectivity of the function space construction.

Proposition 13. *Let D be a continuous cpo, E a consistently complete cpo, and let $a_1, \ldots, a_n \in D$ and $b_1, \ldots, b_n \in E$. Then*

$$\{\langle a_1; b_1\rangle, \ldots, \langle a_n; b_n\rangle\} \textit{ is consistent in } [D \to E]$$

if, and only if,

$$\forall I \subseteq \{1, \ldots, n\} \left(\bigcap_{i \in I} \twoheaduparrow a_i \neq \emptyset \Longrightarrow \{b_i : i \in I\} \textit{ consistent} \right).$$

Proof. For the non-trivial direction define $h: D \to E$ by

$$h(x) = \bigsqcup\{b_i : a_i \ll x\}.$$

Then h is well-defined by consistent completeness and h is monotone. Suppose $A \subseteq D$ is directed and $a_i \ll \sqcup A$. Then there is $d_i \in A$ such that $a_i \ll d_i$ and hence $b_i \sqsubseteq h(d_i)$. Thus

$$h(\bigsqcup A) = \bigsqcup\{b_i : a_i \ll \bigsqcup A\} \sqsubseteq \bigsqcup h[A].$$

Note that if $\{\langle a_1; b_1\rangle, \ldots, \langle a_n; b_n\rangle\}$ is consistent then the function h in the proof is $\bigsqcup_{i=1}^{n} \langle a_i; b_i\rangle$. □

The following characterisation of the continuous function space is easily obtained by Proposition 10, and as a result we have that **CDOM** is cartesian closed.

Theorem 14. *Let D and E be continuous cpos with bases B_D and B_E. If E is consistently complete then $[D \to E]$ is continuous and consistently complete. A base $B_{[D\to E]}$ for $[D \to E]$ is*

$$\left\{\bigsqcup_{i=1}^{n} \langle a_i; b_i\rangle : a_i \in B_D,\ b_i \in B_E,\ \{\langle a_1; b_1\rangle, \ldots, \langle a_n; b_n\rangle\} \text{ consistent}\right\}.$$

Note that if $a \in D_c$ and $b \in E_c$ then the step function $\langle a; b\rangle$ is compact. It follows that if D and E are consistently complete algebraic cpos then there is a base for $[D \to E]$ consisting only of compact elements. Hence $[D \to E]$ is a consistently complete algebraic cpo. Thus **DOM**, the category of consistently complete algebraic cpos with continuous functions as morphisms, is cartesian closed.

2.4. Numberings and effectivecontinuous cpos

In this section we review some definitions and standard facts about numberings. See [15] for basic definitions and [7,9] for more details. Recall first from recursion theory that a set is *recursively enumerable* (*r.e.*, for short) if it is the empty set or if it is the image of a recursive function. An n-ary relation is r.e. if the n-tuples (coded as natural numbers) form a recursively enumerable set. $\{W_e\}_{e\in\omega}$ denotes the family of recursively enumerable sets in some standard numbering of the recursively enumerable sets. Likewise, K_n denotes the finite set of natural numbers with canonical index n in some standard numbering. We let $\langle , \rangle$ denote a recursive pairing function and let π_0 and π_1 denote the corresponding recursive projection functions.

Let A be a set. A *numbering* of A is a surjective function $\alpha : \omega \to A$. We say that the set A is *numbered* (or *indexed*) by α. Note that all information of A is included in α, but we nevertheless write (A, α). If α and β are two numberings of A and B, then $\alpha \times \beta : \omega \to A \times B$ denotes the *product numbering* of A and B, where $(\alpha \times \beta)(n) = (\alpha(\pi_0(n)), \beta(\pi_1(n)))$. Let $\alpha^n : \omega^n \to A^n$ denote the product numbering of n copies of α.

Let $\alpha_i : \omega \to A_i$ be numberings and let $\alpha = \alpha_1 \times \alpha_2 \times \ldots \alpha_n$ be the corresponding product numbering. An n-ary relation $S \subseteq \prod_{i=1}^{n} A_i$ is *α-semidecidable* if $\alpha^{-1}(S)$ is recursively enumerable, and *α-decidable* if $\alpha^{-1}(S)$ is recursive. Let (A, α) and (B, β) be numbered sets. A function $f : A \to B$ is *(α, β)-computable* if there exists a recursive function $\bar{f} : \omega \to \omega$ that *tracks* f, that is, $f(\alpha(n)) = \beta(\bar{f}(n))$ for all $n \in \omega$.

A numbering α of A is said to be *decidable* if the equality relation on A is decidable. We extend the numbering α a numbering $\tilde{\alpha}$ of $\wp_f(A)$, defined by $\tilde{\alpha}(n) := \alpha[K_n]$. Note that if α is decidable then the relation $\alpha(m) \in \tilde{\alpha}(n)$ is decidable.

We now present a weak version of effectivity for continuous cpos.

Definition 15. Let $D = (D; \sqsubseteq, \bot)$ be a continuous cpo. We say that (D, α) is a weakly effective (continuous) domain if $\alpha : \omega \to B$ is a numbering such that B is a base for D and such that the relation $a \ll b$ is α-semidecidable on B, i.e. $\alpha(n) \ll \alpha(m)$ is *r.e.*

Suppose now that (D, α) and (E, β) are weakly effective domains. Intuitively, we want a continuous function $f \in [D \to E]$ to be effective whenever it for every α-computable input x outputs a β-computable element $f(x)$ in an uniform way. By continuity, this means that we demand the relation $b \ll f(a)$ to be (α, β)-semidecidable.

Definition 16. Let (D, α) and (E, β) be weakly effective domains.

(1) An element $x \in D$ is *α-computable* if the set $\alpha^{-1}(\mathrm{approx}(x))$ is *r.e.* An *r.e.* index for the set $\alpha^{-1}(\mathrm{approx}(x))$ is an index for x.

(2) A continuous function $f : D \to E$ is (α, β)-effective if $\beta(m) \ll f(\alpha(n))$ is an *r.e.* relation. An *r.e.* index for the set $\{\langle m, n\rangle : \beta(m) \ll f(\alpha(n))\}$ is an index for f.
(3) $D_{k,\alpha} := \{x \in D : x \text{ is } \alpha\text{-computable}\}$.

In order to obtain a cartesian closed category we consider a stronger notion of effectivity. We restrict ourselves to consistently complete continuous cpos.

Definition 17. A consistently complete continuous cpo $D = (D; \sqsubseteq, \bot)$ is effective if there is a numbering $\alpha : \omega \to B$ of a base B for D such that the following relations are recursive:
(1) $\alpha(n) \ll \alpha(m)$;
(2) $\exists k(\alpha(m),\ \alpha(n) \sqsubseteq \alpha(k))$; and
(3) $\alpha(m) \sqcup \alpha(n) = \alpha(k)$.
We call (D, α) an effective consistently complete continuous cpo.

Remark 18. Note that for an effective consistently complete continuous cpo we have that
(1) $\alpha(n) \sqsubseteq \alpha(m)$ is decidable;
(2) consistency with respect to the way-below relation is equivalent to consistency with respect to $\sqsubseteq$ for a reduced basis. This is because a basis B is reduced if and only if $\{b\}$ is consistent with respect to the way-below relation, for each $b \in B$.

By restricting Definition 17 to algebraic cpos we obtain a cartesian closed category of effective consistently complete algebraic cpos. This is easily proved with the help of the characterisation of the function space obtained in Proposition 13 and Theorem 14.

3. A cartesian closed category of effective consistently complete continuous cpos

In this section we define a restriction on the bases for continuous cpos that gives us a proper characterisation of the relations $\sqsubseteq$ and $\ll$ on the function space. We apply it to Definition 17 and obtain as a result a cartesian closed category of effective consistently complete continuous cpos.

3.1. Almost algebraic bases

In this section we explore a natural criterion on a basis that makes the two basic relations on the function space characterisable in terms of the relations on the base domains.

Definition 19. Let $D = (D; \sqsubseteq, \bot)$ be a continuous cpo. A base B of D is called almost algebraic if the following hold:
(1) For each $a \in B$ there is a sequence $(a_n)_{n\in\omega} \subseteq B$ such that

$$a_0 \gg a_1 \gg \cdots \gg a_n \gg a_{n+1} \gg \cdots \gg a$$

and such that for each $b \in B$, if $b \gg a$ then there is $n \in \omega$ such that $b \gg a_n$.
(2) For each $a, b \in B$, if $\Uparrow a \subseteq \Uparrow b$ then $b \sqsubseteq a$.
We say that D is an almost algebraic cpo if D has an almost algebraic basis. A sequence as in (1) above is called an almost algebraic sequence.

Remark 20. Note the following in connection with Definition 19.
(1) The conditions hold for the basis of compact elements of an algebraic cpo.
(2) An almost algebraic basis B is reduced. This follows from the existence of an almost algebraic sequence.
(3) The chain condition for an almost algebraic sequence can be weakened to only require the existence of a sequence $(a_n)_{n\in\omega} \subseteq B$ with $a_0 \sqsupseteq a_1 \sqsupseteq \cdots \gg a$ such that if $b \gg a$ then there is $n \in \omega$ such that $b \sqsupseteq a_n$, since this can be proven to be equivalent with the given stronger variant by using the interpolation property to construct a chain decreasing with respect to the way-below relation.

Example 21. Two natural examples of non-algebraic cpos that have countable almost algebraic bases are the unit interval domain $\mathbb{I} = ([0, 1]; \leqslant, 0)$ and the continuous interval domain $\mathbb{CIR}$ consisting of closed finite intervals of $\mathbb{R}$. An almost algebraic base for $\mathbb{I}$ is $[0, 1) \cap \mathbb{Q}$, and an almost algebraic base for $\mathbb{CIR}$ is $\{[a, b] : a < b, a, b \in \mathbb{Q}\}$.

The following proposition gives a characterisation of the basic relations on the function space.

Proposition 22. *Let D and E be continuous cpos and suppose that D has an almost algebraic basis B_D and that E has a countable basis B_E. Let $a, c \in B_D$ and $b, d \in B_E$, where $b \neq \perp$. Then the following hold*:
(1) $\langle a; b\rangle \sqsubseteq \langle c; d\rangle \Leftrightarrow c \sqsubseteq a \wedge b \sqsubseteq d$.
(2) $\langle a; b\rangle \ll \langle c; d\rangle \Leftrightarrow c \ll a \wedge b \ll d$.

Proof. Note that the directions $\Leftarrow$ trivially hold without any extra condition. We now show the directions $\Rightarrow$. For (1) we note that $\langle a; b\rangle \sqsubseteq \langle c; d\rangle$ holds if and only if $(\forall x \in D)\ (x \gg a \Rightarrow x \gg c) \wedge b \sqsubseteq d$, since $b \neq \perp_E$. By almost algebraic (2), this is equivalent to $c \sqsubseteq a \wedge b \sqsubseteq d$.

For (2) we let $\langle a; b\rangle \ll \langle c; d\rangle$. Note that $\Uparrow a \neq \emptyset \Rightarrow \Uparrow c \neq \emptyset$, since $b \neq \perp$. Let $(c_i) \gg c$ be a decreasing sequence as in the first condition of almost algebraic and let $(d_i) \ll d$ be an increasing sequence from B_D such that $\bigsqcup_i d_i = d$. From the if direction of (2) we have for each $i \in \omega$,

$$\langle c_i; d_i\rangle \ll \langle c_{i+1}; d_{i+1}\rangle \ll \langle c; d\rangle.$$

We claim that $\bigsqcup_i \langle c_i; d_i\rangle = \langle c; d\rangle$. Note that one direction is immediate.

Let $x \gg c$ and choose an element $c_i \ll x$. Note that $\langle c_i; d_i\rangle(x) = d_i$ and that this holds for almost all i. Hence,

$$\bigsqcup_i \langle c_i; d_i\rangle(x) = \bigsqcup_i d_i = d = \langle c; d\rangle(x).$$

Then for all $x \gg c$

$$\bigsqcup_i \langle c_i; d_i\rangle(x) = \langle c; d\rangle(x).$$

Hence $\langle a; b\rangle \sqsubseteq \langle c_i; d_i\rangle$ holds for all large i and then $c_i \sqsubseteq a \wedge b \sqsubseteq d_i$ follows from (1). From the assumptions we can thus conclude $c \ll c_i \sqsubseteq a$ and $b \sqsubseteq d_i \ll d$. $\square$

3.2. A crucial lemma

In this section we generalise to arbitrary continuous functions the characterisation given in Proposition 22 of the way-below relation on the function space, under the extra assumption that the cpos are consistently complete. This lemma will be crucial in the following when we want to lift the effectivity of the order relations to the function space.

Lemma 23. *Let D and E be consistently complete continuous cpos with bases B_D and B_E. Suppose that B_D is almost algebraic and that B_E is countable. Let $f \in [D \to E]$, let $a \in B_D$ and let $b \in B_E$. Then $\langle a; b\rangle \ll f \Leftrightarrow b \ll f(a)$.*

Proof. The implication from right to left follows from Proposition 10.

We now show the implication from left to right. It suffices to prove it in the case when $f \in B_{[D\to E]}$. Let $f = \bigsqcup_{i\in I} \langle c_i; d_i\rangle$, where I is finite, $c_i \in B_D$ and $d_i \in B_E$. Without loss of generality we assume $b \neq \perp$.

From the assumption that B_D is almost algebraic we obtain descending sequences (a_j) for a and $(c_i^j)_j$ for each c_i, where $i \in I$. Since E is acontinuous cpo and B_E is countable we choose sequences $(d_i^j)_j$ from B_E increasing with respect to $\ll$ such that $\bigsqcup_j d_i^j = d_i$ for each $i \in I$. From this it follows that $\langle c_i^j; d_i^j\rangle \ll \langle c_i; d_i\rangle$ for all $i, j \in \omega$ and hence

$$\bigsqcup_{i\in I} \langle c_i^j; d_i^j\rangle \ll \bigsqcup_{i\in I} \langle c_i; d_i\rangle$$

for all $j \in \omega$. Furthermore, we have

$$\bigsqcup_{i\in I} \langle c_i^j; d_i^j\rangle \ll \bigsqcup_{i\in I} \langle c_i^{j'}; d_i^{j'}\rangle$$

for $j < j'$. The crucial thing to prove is the equality

$$\bigsqcup_{j\in\omega}\bigsqcup_{i\in I}\langle c_i^j; d_i^j\rangle = \bigsqcup_{i\in I}\langle c_i; d_i\rangle \ (*).$$

Let $e \in B_D$, let $d = f(e) = \bigsqcup\{d_i : c_i \ll e\}$ and let $I_0 = \{i : c_i \ll e\}$. The interesting case is when I_0 is non-empty and hence we let $i \in I_0$. Thus we have $c_i^j \ll e$, for sufficiently large j. Let j_0 be such that

$$(j \geqslant j_0) \wedge (i \in I_0) \Rightarrow c_i^j \ll e.$$

Thus for $j \geqslant j_0$ we have $\bigsqcup\{d_i^j : i \in I_0\} \sqsubseteq d$. Let $y \in E$ be such that $y \sqsupseteq d_i^j$, for each $j \geqslant j_0$ and $i \in I_0$. Then $y \sqsupseteq d_i$ for each $i \in I_0$. Hence

$$\bigsqcup_{j\geqslant j_0}\bigsqcup\{d_i^j : i \in I_0\} = d,$$

which proves the equality $(*)$.

From the assumption $\langle a; b\rangle \ll \bigsqcup_{i\in I}\langle c_i; d_i\rangle$ we obtain a j_0 such that if $j \geqslant j_0$ then $\langle a; b\rangle \ll \bigsqcup_I \langle c_i^j; d_i^j\rangle$.

For each a_n in the descending sequence towards a we have

$$\begin{aligned} b &= \langle a; b\rangle(a_n) \\ &\sqsubseteq (\bigsqcup\langle c_i^{j_0}; d_i^{j_0}\rangle)(a_n) \\ &= \bigsqcup\{d_i^{j_0} : c_i^{j_0} \ll a_n\}. \end{aligned}$$

Let $I_n^j = \{i : c_i^j \ll a_n\}$, for each $j \geqslant j_0$. Note that $I_n^j \supseteq I_{n+1}^j$. Hence there exists n_0 such that if $n \geqslant n_0$ then $I_n^j = I_{n_0}^j$ holds, for each $j \geqslant j_0$.

Fix $j \geqslant j_0$. From the assumption $b \neq \perp$ it follows that $I_{n_0}^j \neq \emptyset$. Let $i \in I_{n_0}^j$. Note that $\Uparrow a \subseteq \Uparrow c_i^j$, since if $x \gg a$ then $x \gg a_n$ for all sufficiently large n. Hence $x \gg c_i^j$. By the assumption that B_D is almost algebraic (2) we have $c_i^j \sqsubseteq a$ and thus it follows that $c_i^{j+1} \ll a$.

This implies that

$$\left(\bigsqcup_I \langle c_i^j; d_i^j\rangle\right)(a) = \bigsqcup\{d_i^j : i \in I_{n_0}^j\}.$$

On the other hand,

$$b = \langle a; b\rangle(a_{n_0}) \sqsubseteq \bigsqcup\{d_i^j : i \in I_{n_0}^j\}.$$

From this we obtain $b \ll \bigsqcup\{d_i^k : i \in I_{n_0}^k\}$, for each $k > j$. Note that $c_i \ll a$ holds, since $c_i^k \ll a$. Thus it follows that

$$\begin{aligned} \bigsqcup_{i\in I}\langle c_i; d_i\rangle(a) &= \bigsqcup\{d_i : c_i \ll a\} \\ &\sqsupseteq \bigsqcup\{d_i : i \in I_{n_0}^k\} \\ &\sqsupseteq \bigsqcup\{d_i^k : i \in I_{n_0}^k\} \\ &\gg b. \qquad \square \end{aligned}$$

3.3. Cartesian closure

In this section we prove the main theorem of the section. We start by showing that the natural basis $B_{[D\to E]}$ for $[D \to E]$ is closed and almost algebraic.

Lemma 24. *Let D and E be consistently complete continuous cpos with countable closed and almost algebraic bases B_D and B_E and let $B_{[D\to E]}$ be the basis for $[D\to E]$ obtained as in Theorem 14. Then $B_{[D\to E]}$ is a closed, countable and almost algebraic basis for $[D\to E]$.*

Proof. Note first that $B_{[D\to E]}$ is trivially closed when B_D and B_E are closed. Recall from Remark 18 that consistency with respect to the way-below relation is equivalent to consistency with respect to $\sqsubseteq$ for an almost algebraic (and hence reduced) basis.

Let I be a finite set and let

$$f = \bigsqcup\{\langle a_i; b_i\rangle : a_i \in B_D, b_i \in B_E, i \in I\} \in B_{[D\to E]}.$$

Thus the set $\{\langle a_i; b_i\rangle : a_i \in B_D,\ b_i \in B_E,\ i \in I\}$ is consistent. For each $i \in I$, let $a_i^j \ll a_i$ be a sequence from B_D, which is increasing with respect to $\ll$ such that $\bigsqcup\{a_i^j : j \in \omega\} = a_i$ and let $b_i^j \gg b_i$ be an almost algebraic sequence for b_i. We need to show that there is $j \in \omega$ such that

$$\{\langle a_i^j; b_i^j\rangle : a_i^j \in B_D,\ b_i^j \in B_E,\ i \in I\}$$

is consistent.

Fix $J \subseteq I$ and let $A_J = \{a_i : i \in J\}$. Suppose that A_J is consistent. Then $C_J = \{b_i : i \in J\}$ is consistent by Proposition 13 and hence $\bigsqcup C_J \in B_E$, since B_E is closed. Then there is $\bar{b} \in B_E$ such that $\bar{b} \gg C_J$, since B_E is almost algebraic. It follows that there is $j_J \in \omega$ such that for each $j \geqslant j_J$ we have $b_i^j \ll \bar{b}$ and hence that $\{b_i^j : i \in J\}$ is a consistent set in B_E, for each $j \geqslant j_J$.

On the other hand, consider the case when A_J is inconsistent. Suppose that $A_J^j = \{a_i^j : i \in J\}$ is consistent, for each $j \in \omega$. We will show that this results in a contradiction. Note that $\bigsqcup A_J^j$ exists for each $j \in \omega$ and hence $A = \{\bigsqcup A_J^j : j \in \omega\}$ is an increasing chain in B_D. Thus $\bigsqcup A$ exists and $\bigsqcup A \sqsupseteq A_J$, since if $x \ll a_i \in A_J$ then there is $j \in \omega$ such that $x \ll a_i^j \in A_J^j$. This shows that A_J is consistent and thus there is $j_J \in \omega$ such that for each $j \geqslant j_J$ we have that A_J^j is inconsistent.

Choose $j \in \omega$ such that $j \geqslant j_J$, for each of the finitely many $J \subseteq I$. Suppose that $\{a_i^j : i \in J\}$ is consistent. Then A_J is consistent by construction and hence $\{b_i^j : i \in J\}$ is consistent. This shows that

$$\{\langle a_i^k; b_i^k\rangle : a_i^k \in B_D,\ b_i^k \in B_E,\ i \in I\}$$

is consistent for each $k \geqslant j$. It follows that

$$\{\bigsqcup\{\langle a_i^k; b_i^k\rangle : a_i^k \in B_D,\ b_i^k \in B_E,\ i \in I\} : k \geqslant j\}$$

is an almost algebraic sequence for f.

We now show that $B_{[D\to E]}$ is almost algebraic (2). It suffices to show that

$$\Uparrow \bigsqcup_I \langle c_i; d_i\rangle \subseteq \Uparrow \langle a; b\rangle \Rightarrow \langle a; b\rangle \sqsubseteq \bigsqcup_I \langle c_i; d_i\rangle$$

for a finite set I and $b \neq \perp$. Suppose that $\Uparrow\bigsqcup_I \langle c_i; d_i\rangle \subseteq \Uparrow\langle a; b\rangle$. Since B_E is an almost algebraic basis we have almost algebraic sequences $d_i^n \gg d_i$. By continuity we choose increasing sequences $c_i^n \ll c_i$ in B_D such that $\bigsqcup_{n\in\omega} c_i^n = c_i$. Hence $\langle c_i; d_i\rangle \ll \langle c_i^n; d_i^n\rangle$ and we conclude that $\bigsqcup_I \langle c_i; d_i\rangle \ll \bigsqcup_I \langle c_i^n; d_i^n\rangle$ holds for all large n such that the supremum exists.

By assumption we have $\langle a; b\rangle \ll \bigsqcup_I \langle c_i^n; d_i^n\rangle$ and hence $b \ll \bigsqcup_I\{d_i^n : c_i^n \ll a\}$ follows by Lemma 23. We let $J^n = \{i \in I : c_i^n \ll a\}$. Recall that $b \neq \perp$ implies $J^n \neq \emptyset$. But J^n is finite and hence there must be an index n_0 such that $J^n = J^{n_0}$ for each $n \geqslant n_0$. Let $J = J^{n_0}$. Hence $c_i \sqsubseteq a$ holds for all $i \in J$ and then it follows that $\bigsqcup_J c_i \sqsubseteq a$ and $b \ll \bigsqcup_J d_i^n$, for each sufficiently large n. It remains to show that $b \sqsubseteq \bigsqcup_J d_i$. Let $y \in E$. We have

$$y \gg \bigsqcup_J d_i \Rightarrow y \gg \bigsqcup_J d_i^n$$

for sufficiently large n. Hence $y \gg b$ holds and then $b \sqsubseteq \bigsqcup_J d_i$ follows from the assumption that B_E is almost algebraic (2). Let $x \gg a$. Then $x \gg c_i$ for each $i \in J$ and hence

$$\bigsqcup_I \langle c_i; d_i \rangle(x) \sqsupseteq \bigsqcup_J d_i \sqsupseteq b = \langle a; b \rangle(x).$$

It follows that $\langle a; b \rangle \sqsubseteq \bigsqcup_I \langle c_i; d_i \rangle$. $\square$

The following theorem is an immediate consequence of Lemma 24.

Theorem 25. *The category of consistently complete continuous cpos that have a countable, closed and almost algebraic basis, with continuous functions as morphisms, is cartesian closed.*

We now consider effectivity.

Theorem 26. *Let (D, α) and (E, β) be effective consistently complete continuous cpos and let $\alpha[\omega] = B_D$ and $\beta[\omega] = B_E$ be closed and almost algebraic bases for D and E, respectively. Then there is a numbering $\gamma : \omega \to B_{[D\to E]}$, obtained uniformly from α and β, such that $([D \to E], \gamma)$ is an effective consistently complete continuous cpo.*

Proof. A numbering $\gamma : \omega \to B_{[D\to E]}$ is constructed as follows. Let N be the set of indices for finite subsets of step functions given by Theorem 14, i.e. let

$$N = \{n : \{\langle \alpha(i); \beta(j) \rangle : \langle i, j \rangle \in K_n\} \text{ consistent}\}.$$

Then γ is defined by

$$\gamma(n) = \begin{cases} \bigsqcup \{\langle \alpha(i); \beta(j) \rangle : \langle i, j \rangle \in K_n\} & \text{if } n \in N, \\ \perp & \text{otherwise.} \end{cases}$$

Note that N is recursive set. This is because membership in N is decided by deciding the consistency of a finite set of step functions, which is decidable by Proposition 13 and the assumptions.

We take the liberty to suppress the numberings in the rest of the proof. We first show that $\sqsubseteq$ on $B_{[D\to E]}$ is γ-decidable. Note that it is sufficient to show

$$\langle a; b \rangle \sqsubseteq \bigsqcup_I \langle c_i; d_i \rangle \Leftrightarrow b = \perp_E \vee (\exists J \subseteq I) \left(\bigsqcup_J c_i \sqsubseteq a \wedge b \sqsubseteq \bigsqcup_J d_i \right), \quad (*)$$

for a finite set of step functions indexed by I.

The direction right to left is immediate. The direction left to right holds because B_D is almost algebraic. To prove this we let $(a_n) \gg a$ be an almost algebraic sequence for a and assume that $b \neq \perp_E$. Let $x \in D$ and suppose that $a \ll x$ holds. Then we know firstly that there is some $a_n \ll x$ and secondly that

$$C^n := \{c_i : i \in I \wedge c_i \ll a_n\} \neq \emptyset.$$

This holds since

$$\langle a; b \rangle(a_n) = b \sqsubseteq \bigsqcup_I \langle c_i; d_i \rangle(a_n),$$

and hence $C^n \neq \emptyset$, since $b \neq \perp_E$. Since I is finite there must be an n_0 such that for all $n \geqslant n_0$ we have $C^n = C^{n+1}$. Let $J = \{i \in I : c_i \ll a_{n_0}\}$. By almost algebraic (2) we know that $\bigsqcup_J c_i \sqsubseteq a$ and hence that $\bigsqcup_J d_i \sqsupseteq b$ holds. This proves the equivalence $(*)$ and thus we have shown that $\sqsubseteq$ is γ-decidable.

We now show that $([D \to E], \gamma)$ is an effective consistently complete continuous cpo. Let $h = \bigsqcup_J \langle a_j; b_j \rangle$ and $g = \bigsqcup_I \langle c_i; d_i \rangle$ be elements of $B_{[D\to E]}$, where J and I are finite sets. We first show Definition 17(1). We need to show that the relation $h \ll g$ is γ-decidable. Note that

$$\bigsqcup_J \langle a_j; b_j \rangle \ll \bigsqcup_I \langle c_i; d_i \rangle \Leftrightarrow (\forall j \in J) \left(\langle a_j; b_j \rangle \ll \bigsqcup_I \langle c_i; d_i \rangle \right).$$

The right-hand side is equivalent to

$$(\forall j \in J)\left(b_j \ll \bigsqcup_I \{d_i : a_j \ll c_i\}\right)$$

by Lemma 23, and this expression is decidable since D and E are effective consistently complete continuous cpos.

To show Definition 17(2) we note that h and g are consistent if and only if the set

$$L = \{\langle a_j; b_j\rangle : j \in J\} \cup \{\langle c_i; d_i\rangle : i \in I\}$$

is consistent. Checking the consistency of L reduces by Proposition 13 to checking the consistency of finite sets of elements in B_D and B_E, and this is decidable by assumption.

We now show Definition 17(3). Suppose that h and g are consistent. For $f \in B_{[D\to E]}$ we have that $f = g \sqcup h$ if and only if $f = \bigsqcup\{\langle a; b\rangle \in L\}$. This is recursive, since $\sqsubseteq$ and then $=$ is γ-decidable on $B_{[D\to E]}$. Thus $([D\to E], \gamma)$ is an effective consistently complete continuous cpo. □

Note that for $f \in [D\to E]$ we have the following equivalences:

$$\bigsqcup_I \langle c_i; d_i\rangle \ll f \Leftrightarrow (\forall i \in I)(\langle c_i; d_i\rangle \ll f)$$

$$\Leftrightarrow (\forall i \in I)(d_i \ll f(c_i)),$$

where the last one holds by Lemma 23. Hence f is (α, β)-effective if and only if f is γ-computable.

It follows that the category of effective consistently complete continuous cpos with closed and almost algebraic bases is cartesian closed.

Theorem 27. *The category of effective consistently complete continuous cpos with closed and almost algebraic bases and effective continuous functions as morphisms is cartesian closed.*

Proof. Let (D, α) and (E, β) be effective consistently complete continuous cpos with closed and almost algebraic bases B_D and B_E, respectively. The exponent $([D\to E], \gamma)$ is an effective consistently complete continuous cpo by Theorem 26, with a closed and almost algebraic basis $B_{[D\to E]}$ by Lemma 24. □

4. C-bifinite domains

In this section we extend some of our results to a category of continuous cpos containing objects that are not consistently complete.

4.1. Basic definitions

In this section we generalise the definitions and results for the category of effective bifinite domains in [11], and we refer to that paper for more details. We make a generalisation of the definition of a complete set, demanding the sets to be complete with respect to the way-below relation. The lemmas and definitions that follow are all straightforward generalisations of the bifinite case.

Definition 28. Let $(D; \sqsubseteq, \bot)$ be a cpo. Given $B \subseteq D$, we say that B is a way-above-complete (wa-complete) set (in D) if

$$\forall C \subseteq B\, \forall x \gg C\, \exists b \in B\, (x \gg b \sqsupseteq C).$$

Lemma 29. *Let D be a cpo and let $A \subseteq D$. Then A is wa-complete if and only if the maximum of $A \cap \twoheaddownarrow x$ exists in A for each $x \in D$.*

Definition 30. Let $(D; \sqsubseteq, \bot)$ be a cpo and let $B \subseteq D$. We call a family $\mathcal{F} = \{B_i : i \in I\}$ of finite subsets of B a way-above-complete cover of B if each B_i is wa-complete and for each $A \subseteq_f B$ there is $i \in I$ such that $A \subseteq B_i$.

Definition 31. Let $(D; \sqsubseteq, \perp)$ be a continuous cpo. We say that D is a c-bifinite domain if there is a basis B of D that has a wa-complete cover. Then B is called a c-bifinite basis for D.

Remark 32. Note that each consistently complete continuous cpo is c-bifinite. Furthermore, note that every algebraic cpo with a c-bifinite basis is bifinite.

Definition 33. Let D and E be continuous cpos with bases B_D and B_E.
(1) A set $u = \{(a_i, b_i) : i \in I\} \in \wp^*_f(B_D \times B_E)$ is c-joinable if the set $A := \{a_i : i \in I\}$ is wa-complete and $a_i \sqsubseteq a_j \Rightarrow b_i \sqsubseteq b_j$.
(2) If u is c-joinable then we define the function $s_u : D \to E$ by

$$s_u(x) = b_i \;\Leftrightarrow\; a_i = \max(A \cap \mathop{\Downarrow} x).$$

Remark 34. Note that s_u above is well-defined, since A is wa-complete.

The following lemma is an essential ingredient in the function space construction.

Lemma 35. *Let D and E be c-bifinite domains with c-bifinite bases B_D and B_E, and let $u \subseteq_f B_D \times B_E$ be c-joinable.*
(1) *s_u is continuous.*
(2) $s_u = \bigsqcup\{\langle a; b\rangle : (a, b) \in u\}$.
(3) $B_{[D\to E]} := \{s_u : u \in \wp^*_f(B_D \times B_E) \wedge u$ c-joinable$\}$ *is a basis of* $[D \to E]$.

Proof. (1) is easy. For (2) let $(a, b) \in u$ and $\pi_0(u) = A$. Clearly we have $\langle a; b\rangle \sqsubseteq s_u$, for each $(a, b) \in u$. Let $f \in [D \to E]$ be such that $\langle a; b\rangle \sqsubseteq f$, for each $(a, b) \in u$. Then $s_u(x) \sqsubseteq f(x)$ holds for all $x \in D$, since $s_u(x)$ is decided by the value of one step function in u. This shows (2).

The argument for (3) is more involved. Let $f \in [D \to E]$. By Proposition 10 it is sufficient to show that

$$\mathcal{F} := \{s_u : u \text{ c-joinable} \wedge (\forall (a, b) \in u)\, (b \ll f(a))\}$$

is a directed set such that $f = \bigsqcup \mathcal{F}$. Let $s_u, s_v \in \mathcal{F}$, let $A_u = \pi_0(u)$ and let $A_v = \pi_0(v)$. Since D is a c-bifinite domain we can find a finite wa-complete $\bar{A}$ such that $\bar{A} \supseteq A_u \cup A_v$. Likewise, since E is a c-bifinite domain we can find a finite wa-complete $\bar{B}$ such that $\bar{B} \supseteq \pi_1(u) \cup \pi_1(v)$. For each $x \in D$ we let $a_x = \max(\bar{A} \cap \mathop{\Downarrow} x)$ and $b_x = \max(\bar{B} \cap \mathop{\Downarrow} f(a_x))$. We now claim that $w := \{(a_x, b_x) : x \in D\}$ is c-joinable and that $s_w \in \mathcal{F}$ is such that $s_u, s_v \sqsubseteq s_w$.

In order to prove that w is c-joinable we first note that $A_w = \pi_0(w)$ is wa-complete by construction. Note further that if $a_x \sqsubseteq a_y \in A_w$ then $b_x \sqsubseteq b_y$ follows from the monotonicity of f. Furthermore, $b_x \ll f(a_x)$ holds and thus we have shown that $s_w \in \mathcal{F}$.

We now show $s_u \sqsubseteq s_w$. Let $(a, b) \in u$ and let $x \gg a$. Then $a_x \sqsupseteq a$ and thus $f(a_x) \sqsupseteq f(a) \gg b$. Since b_x is the maximal element in $\bar{B}$ way-below $f(a_x)$ we have

$$s_w(x) = b_x \sqsupseteq b = \langle a; b\rangle(x).$$

It follows that $\langle a; b\rangle \sqsubseteq s_w$ and then $s_u \sqsubseteq s_w$. We prove $s_v \sqsubseteq s_w$ in a similar way and thus we have shown that $\mathcal{F}$ is directed.

It remains to show that $f = \bigsqcup \mathcal{F}$. Fix $x \in D$ and let $b \ll f(x)$. By continuity we choose $a \ll x$ such that $b \ll f(a)$. We interpolate and find an element b' such that $b \ll b' \ll f(a)$, and hence we have that $\langle a; b'\rangle \in \mathcal{F}$. (Actually, $\langle a; b'\rangle$ is not c-joinable, but it is an element of some c-joinable set of step functions in $\mathcal{F}$. See Remark 36 following directly after the proof.) Thus $b \ll \bigsqcup \mathcal{F}(x)$, and we conclude that $f(x) \sqsubseteq \bigsqcup \mathcal{F}(x)$. The other direction is immediate. This shows (3). □

Remark 36. We see that suprema of finite consistent sets of step functions built from c-joinable sets form a base. Especially, to each step function $\langle a; b\rangle$ (for $a \neq \perp$) there is a corresponding c-joinable set $\{(\perp, \perp), (a, b)\}$.

We now relate the c-bifinite domains to the cartesian closed category of retracts of bifinite domains. Recall first the notion of a retract.

Definition 37. Let D and E be cpos. A pair $s : D \to E$ and $r : E \to D$ of continuous functions is called a continuous section retraction pair if $r \circ s = \mathrm{id}_D$. The image D of E under r is called a retract of E.

A *deflation* is a continuous function on a cpo, which is below the identity function and which has a finite image. (See [12,13] for details.) The following result is Theorem 4.1 of [13].

Theorem 38. *A continuous cpo $D = (D; \sqsubseteq, \bot)$ is a retract of a bifinite domain if and only if there is a directed family $\mathcal{F}$ of deflations on D such that $\bigsqcup \mathcal{F} = \mathrm{id}_D$.*

With this theorem the following result easily follows.

Proposition 39. *If D is a c-bifinite domain then D is a retract of a bifinite domain.*

Proof. Let D be a continuous cpo, let B be a basis for D and let $N \subseteq_f B$ be a wa-complete set. Define the function $P_N : D \to D$ by

$$P_N(x) = \max(N \cap \mathop{\downarrow\!\!\downarrow} x),$$

for each $x \in D$. Note that P_N is a deflation. Suppose that $\mathcal{F}$ is a wa-complete cover of D. We claim that

$$\mathcal{G} := \{P_N : N \in \mathcal{F}\}$$

is a directed family of deflations on D such that $\bigsqcup \mathcal{G} = \mathrm{id}_D$. To see that it is directed, let $P_{N_0}, P_{N_1} \in \mathcal{G}$ and let $N \in \mathcal{F}$ be such that $N \supseteq N_0, N_1$. Then $P_N \in \mathcal{G}$ is such that $P_N \sqsupseteq P_{N_0}, P_{N_1}$.

Clearly we have that $\bigsqcup \mathcal{G} \sqsubseteq \mathrm{id}_D$. Fix $x \in D$ and let $b \in B$ be such that $b \ll x$. It suffices to show that $\bigsqcup \mathcal{G}(x) \sqsupseteq b$. Note that there is $N \in \mathcal{F}$ such that $b \in N$, since $\mathcal{F}$ is a wa-complete cover of B. It follows that $P_N(x) \sqsupseteq b$ and hence that $\bigsqcup \mathcal{G}(x) = x$. $\square$

4.2. A crucial lemma revisited

The purpose of this section is to prove the analogue of Lemma 23 for c-bifinite domains. We first need the following technical lemma, showing that we can modify given approximating sequences so as to reflect relations between elements in a stricter way.

Lemma 40. *Let D be a continuous cpo, let $V = \{x_i : i \in I\}$ be a finite subset of D and let B be countable base of D. Then there are sequences $(d_i^j)_j$ in B such that*

(1) $d_i^j \ll d_i^{j+1}$ *for all* $i \in I$, $j \in \omega$.

(2) $\bigsqcup_j d_i^j = x_i$ *for all* $i \in I$.

(3) $x_i \sqsubset x_{i'} \Rightarrow d_i^j \ll d_{i'}^j$ *for all* $i, i' \in I$, $j \in \omega$.

Proof. For each $i \in I$, let $(\bar{d}_i^j)_j$ be sequences in B satisfying (1) and (2). These exist by the assumption that B is countable.

Let $V_0 = \{x \in V : x \text{ is } \sqsubseteq\text{-minimal in } V\}$. For the inductive step we let

$$V_{k+1} = \{x \in V \setminus \bigcup_{l \leqslant k} V_l : x \text{ is } \sqsubseteq\text{-minimal in } V \setminus \bigcup_{l \leqslant k} V_l\}.$$

This gives for all k and $x, y \in V_k$ that if $x \sqsubseteq y$ then $x = y$.

For $x_i \in V_0$ we let $d_i^j = \bar{d}_i^j$, for each $j \in \omega$. Now suppose that d_i^j have been defined for all j and i such that $x_i \in \bigcup_{l \leqslant k} V_l$, and suppose (1)–(3) hold for these.

Fix j and suppose that $x_i \in V_{k+1}$. Note that if $x_t \sqsubset x_i$ then $x_t \in \bigcup_{l \leqslant k} V_l$ and note that there are only finitely many such x_t. Thus there is $\bar{d}_i^{j_t}$ such that $d_t^j \ll \bar{d}_i^{j_t}$, for each $x_t \sqsubset x_i$. Then choose $d_i^j = \bar{d}_i^s$, where $\bar{d}_i^s \gg d_i^{j-1}$ if $j > 0$ and $\bar{d}_i^s \gg \bar{d}_i^{j_t}$, for each $x_t \sqsubset x_i$. For the resulting sequence $(d_i^j)_j$ we have that (3) holds. $\square$

Here again is a crucial lemma.

Lemma 41. *Let D and E be c-bifinite domains with c-bifinite bases B_D and B_E. Suppose that B_D is almost algebraic and that B_E is countable. Furthermore, let $f \in [D \to E]$, $a \in B_D$ and $b \in B_E$. Then $\langle a; b\rangle \ll f \Leftrightarrow b \ll f(a)$.*

Proof. The if direction holds in every cpo. We can without loss of generality assume $b \neq \perp$. For the only if direction it suffices by Lemma 35 to show

$$\langle a; b\rangle \ll s_u \;\Leftrightarrow\; b \ll s_u(a)$$

for every c-joinable u. Let $u = \{(c_i, d_i)\}_{i\in I}$ be c-joinable, let $\langle a; b\rangle \ll s_u$ and let $C = \{c_i : i \in I\}$. We prove that $b \ll d_i$ holds for $c_i = \max(C \cap \downarrow\!\!\!\!\downarrow a)$, showing that $b \ll s_u(a)$.

The first goal is to construct a particular sequence of c-joinable sets v_k such that $s_{v_k} \ll s_{v_{k+1}} \ll s_u$ for each k, and such that $s_u = \bigsqcup_k s_{v_k}$. For each $i \in I$ we let $(c_i^j)_j$ be an almost algebraic sequence for c_i. Then let $(d_i^j)_j$ be sequences obtained as in Lemma 40 for the set $\{d_i : i \in I\}$. In particular, $d_i = \bigsqcup_j d_i^j$ holds together with the implications

$$c_i \sqsubseteq c_{i'} \Rightarrow d_i \sqsubseteq d_{i'} \Rightarrow (\forall j \in \omega)(d_i^j \sqsubseteq d_{i'}^j)$$

for each $i, i' \in \omega$. For each $j \in \omega$ we let $C_j = \{c_i^j : i \in I\}$. Let $\bar{C}_0 \supseteq C_0$ be a finite and wa-complete set obtained from the wa-complete cover of B_D. For each $e \in \bar{C}_0$ we let $I_0^e = \{i \in I : c_i \ll e\}$. Let $j_0^e \in \omega$ be such that for all i in I_0^e we have that $c_i^{j_0^e} \ll e$ and choose $j_0 \in \omega$ such that $(\forall e \in \bar{C}_0)(j_0 \geqslant j_0^e)$. This is possible since the sequences are almost algebraic and I and $\bar{C}_0$ are finite sets. Note that it follows for each e that

$$j \geqslant j_0 \wedge e \in \bar{C}_0 \wedge e \gg c_i \Rightarrow e \gg c_i^j.$$

Suppose now for the inductive step that $\bar{C}_k$ and j_k have been defined such that $j_k \geqslant j_{k-1}$ and such that the following hold:
(1) $\bar{C}_k \supseteq C_{j_{k-1}}$ (with $j_{-1} = 0$).
(2) $(e \in \bar{C}_k) \wedge (e \gg c_i) \Rightarrow e \gg c_i^{j_k}$.
(3) $\bar{C}_k$ is finite and wa-complete.
Let $\bar{C}_{k+1} \supseteq C_{j_k}$ be a finite and wa-complete set obtained from the wa-complete cover and define $I_{k+1}^e = \{i \in I : c_i \ll e\}$ for each $e \in \bar{C}_{k+1}$. Let $j_{k+1}^e \in \omega$ be such that $(\forall i \in I_{k+1}^e)(c_i^{j_{k+1}^e} \ll e)$ and let $j_{k+1} \in \omega$ then be such that $j_{k+1} \geqslant j_k$ and $(\forall e \in \bar{C}_{k+1})(j_{k+1} \geqslant j_{k+1}^e)$. It is clear that this construction gives a sequence of $\bar{C}_k$ and j_k satisfying (1)–(3) above, for each $k \in \omega$.

We now construct a c-joinable set v_k for each pair $(\bar{C}_k, j_k)$ of the sequence. For each $e \in \bar{C}_k$ we let $c_{i_e} = \max(C \cap \downarrow\!\!\!\!\downarrow e)$ and let $v_k = \{(e, d_{i_e}^k) : e \in \bar{C}_k\}$. Suppose that $e \sqsubseteq e' \in \bar{C}_k$. Then $c_{i_e} \sqsubseteq c_{i_{e'}}$ and hence $d_{i_e} \sqsubseteq d_{i_{e'}}$ holds, since u is assumed to be c-joinable. Then $d_{i_e}^k \sqsubseteq d_{i_{e'}}^k$ holds by Lemma 40 and thus we conclude that v_k is c-joinable, since $\bar{C}_k$ is wa-complete.

We now show that $s_{v_k} \ll s_u$. Let $(e, d_{i_e}^k) \in v_k$. Note that

$$\langle e; d_{i_e}^k\rangle \ll \langle c_{i_e}; d_{c_{i_e}}\rangle \sqsubseteq s_u,$$

since $c_{i_e} \ll e$ and $d_{i_e}^k \ll d_{i_e}$ hold by construction. Thus it follows that $s_{v_k} \ll s_u$.

We now show that $s_{v_k} \ll s_{v_{k+1}}$. Let $(e, d_{i_e}^k) \in v_k$. Then $c_{i_e}^{j_k} \ll e$ follows from (ii) above, where $c_{i_e}^{j_k} \in C_{j_k} \subseteq \bar{C}_{k+1}$. It follows that $c_{i_e} = \max(C \cap \downarrow\!\!\!\!\downarrow c_{i_e}^{j_k})$, since $c_{i_e} \ll c_{i_e}^{j_k}$ holds. Thus we have shown $(c_{i_e}^{j_k}, d_{i_e}^{k+1}) \in v_{k+1}$. From the assumption $d_{i_e}^k \ll d_{i_e}^{k+1}$ we then finally deduce

$$\langle e; d_{i_e}^k\rangle \ll \langle c_{i_e}^{j_k}; d_{i_e}^{k+1}\rangle \sqsubseteq s_{v_{k+1}}.$$

This shows $s_{v_k} \ll s_{v_{k+1}}$.

We now show that $\bigsqcup_{k\in\omega} s_{v_k} = s_u$. Let $x \in D$ and let $c_i = \max(C \cap \downarrow\!\!\!\!\downarrow x)$. Then it follows that $s_u(x) = d_i$. By the construction above we obtain a k_0 such that if $k \geqslant k_0$ then $c_i^{j_k} \ll x$. Fix $k \geqslant k_0$. Then $c_i = \max(C \cap \downarrow\!\!\!\!\downarrow c_i^{j_k})$.

Furthermore, $c_i^{j_k} \in \bar{C}_{k+1}$ and hence $(c_i^{j_k}, d_i^{k+1}) \in v_{k+1}$ and $s_{v_{k+1}}(x) = \langle c_i^{j_k}; d_i^{k+1}\rangle(x) = d_i^{k+1}$. But $k \geqslant k_0$ was arbitrary. Thus

$$d_i = \bigsqcup_k d_i^k \sqsubseteq \bigsqcup_k s_{v_k}(x) \sqsubseteq s_u(x) = d_i,$$

showing that $\bigsqcup_{k\in\omega} s_{v_k} = s_u$.

Let $\langle a; b\rangle \ll s_u$. Then there is $k \in \omega$ such that $\langle a; b\rangle \sqsubseteq s_{v_k}$. For an almost algebraic decreasing sequence $(a_n)_n$ for a we let $e_n = \max(\bar{C}_k \cap \Downarrow a_n)$ and $c_{i_n} = \max(C \cap \Downarrow e_n)$. Note that if $n < m$ then $e_m \sqsubseteq e_n$ and $c_{i_m} \sqsubseteq c_{i_n}$. Thus there is some n_0 such that

$$n \geqslant n_0 \Rightarrow e_n = e_{n_0} \wedge c_{i_n} = c_{i_{n_0}}.$$

Let $\bar{e} = e_{n_0}$ and $\bar{c} = c_{i_{n_0}}$. Note that $s_{v_k}(a_n) = d^k_{i_{n_0}}$ holds for each $n \geqslant n_0$ and thus $b \sqsubseteq d^k_{i_{n_0}}$. Note also that $\bar{e} \ll a_n$ holds for each $n \geqslant n_0$ and hence that $\Uparrow a \subseteq \Uparrow \bar{e}$. By the assumption that D is almost algebraic it follows that $\bar{e} \sqsubseteq a$ and thus that $\bar{c} \ll a$. Let $\tilde{c} = \max(C \cap \Downarrow a)$. Then $\bar{c} \sqsubseteq \tilde{c}$ holds and hence $d_{\bar{c}} \sqsubseteq d_{\tilde{c}}$, since u is c-joinable. By Lemma 40 we obtain $d^t_{\bar{c}} \sqsubseteq d^t_{\tilde{c}}$, for each $t \in \omega$ and thus

$$b \sqsubseteq d^k_{i_{n_0}} = d^k_{\bar{c}} \sqsubseteq d^k_{\tilde{c}} \ll d_{\tilde{c}} = s_u(a). \qquad \square$$

4.3. A closure result for the function space

In this section we show that the function space of two c-bifinite domains is a c-bifinite domain, given the assumptions of Lemma 41.

Theorem 42. *Let D and E be c-bifinite domains with c-bifinite bases B_D and B_E, respectively. Suppose that B_D is almost algebraic and that B_E is countable. Then $[D \to E]$ is a c-bifinite domain.*

Proof. By Lemma 35, a base for $[D \to E]$ is

$$B_{[D\to E]} = \{s_u : u \in \wp_f^*(B_D \times B_E) \wedge u \text{ c-joinable}\}.$$

We show that $B_{[D\to E]}$ has a wa-complete cover. It suffices to show that an arbitrary set on the form

$$M = \{s_{u_0}, \ldots, s_{u_k}\} \subseteq B_{[D\to E]}$$

is contained in a finite wa-complete set.

Let $A = \bigcup_0^k \pi_0(u_i)$ and $B = \bigcup_0^k \pi_1(u_i)$. Furthermore, let $\bar{A} \supseteq A$ and $\bar{B} \supseteq B$ be finite wa-complete sets that exist by hypothesis. Then let

$$\bar{M} = \{s_u : u \subseteq \bar{A} \times \bar{B} \text{ and } u \text{ is c-joinable}\}.$$

Clearly we have that $\bar{M}$ is a finite superset of M. It remains to show that $\bar{M}$ is wa-complete. Let $f \in [D \to E]$. We need to show that $\max(\bar{M} \cap \Downarrow f)$ exists. For each $x \in D$ we let $a_x = \max(\bar{A} \cap \Downarrow x)$ and $b_x = \max(\bar{B} \cap \Downarrow f(a_x))$. Then we define the set $w = \{(a_x, b_x) : x \in D\}$ and claim that $s_w = \max(\bar{M} \cap \Downarrow f)$.

We first note that $\{a_x : x \in D\}$ is wa-complete and hence that w is c-joinable and $w \in \bar{M}$, since f is monotone. We also note that $b_x \ll f(a_x)$, from which it follows that $\langle a_x; b_x\rangle \ll f$ by Proposition 10. And since $s_w = \bigsqcup_{x\in D}\langle a_x; b_x\rangle$, it follows that $s_w \ll f$.

We now show that $s_w = \max(\bar{M} \cap \Downarrow f)$. Suppose that $s_v \in \bar{M}$ and that $s_v \ll f$. Let $(a, b) \in v$. Hence it follows that $\langle a; b\rangle \ll f$ and by Lemma 41 we conclude that $b \ll f(a)$.

Let $x \gg a$. Then $a \sqsubseteq a_x \ll x$ and thus it follows that $b \ll f(a) \sqsubseteq f(a_x)$. Hence we have $b \sqsubseteq b_x$ and then

$$\langle a; b\rangle(x) = b \sqsubseteq b_x = \langle a_x; b_x\rangle(x) = s_w(x).$$

It follows that $s_v \sqsubseteq s_w$ and hence we have shown $s_w = \max(\bar{M} \cap \Downarrow f)$. $\square$

It is unfortunately not the case that the canonical basis $B_{[D\to E]}$ for $[D\to E]$ is almost algebraic, given the assumptions that B_D and B_E are countable almost algebraic and c-bifinite bases for D and E. One problem is that $B_{[D\to E]}$ need not even be reduced.

Example 43. Let $E = (E; \sqsubseteq_E, \perp_E)$ be acontinuous cpo with countable, c-bifinite and almost algebraic basis B_E and let $\omega + 1$ and $\overline{\omega + 1}$ be two ω-chains with the natural orderings. Then let $D = (D; \sqsubseteq, \perp)$ be the following c-bifinite domain. The underlying set is $D = E \cup \omega + 1 \cup \overline{\omega + 1}$. The ordering $\sqsubseteq$ is the reflexive and transitive closure of $\sqsubseteq_E$, the orderings on the two ω-chains and the basic inequalities:

$$n \sqsubseteq x \text{ when } n \in \omega + 1 \wedge x \in E; \text{ and}$$

$$n \sqsubseteq \overline{n} \text{ when } n \in \omega + 1 \wedge \overline{n} \in \overline{\omega + 1}.$$

Then $D = (D; \sqsubseteq, \perp)$ is continuous cpo with $\perp = 0$. A basis for D is

$$B_D := B_E \cup \omega + 1 \cup \{\overline{n} : \overline{n} \in \overline{\omega + 1} \wedge \overline{n} \neq \overline{\omega}\}.$$

Note that $\overline{\omega} \notin B_D$. We leave the verification that B_D is a countable, c-bifinite and almost algebraic basis to the reader. (Note that a corresponding result can also be proven if we, for example, substitute two copies of the continuous cpo $[0, 1]$ for the two copies of $\omega + 1$.)

Now let D' be the boolean domain $\{\perp, L, R\}$. We know from Lemma 35 that the set

$$B_{[D\to D']} := \{s_u : u \in \wp_f^*(B_D \times D'_c) \wedge u \text{ c-joinable}\}$$

is a basis for $[D\to D']$. Consider the set $u := \{(0, \perp), (\overline{0}, L), (\omega, R)\}$. Clearly u is c-joinable. Suppose now that $f \in B_{[D\to D']}$ is such that $f \gg s_u$. Then $\langle \overline{0}; L\rangle \ll f$ and $\langle \omega; R\rangle \ll f$ hold. But then $L \ll f(\overline{0})$ and $R \ll f(\omega)$, by Lemma 41, and since f is monotone it follows that $L \ll f(\overline{\omega})$ and $R \ll f(\overline{\omega})$. This is a contradiction, since the elements L and R are not consistent in the boolean domain. It follows that the basis $B_{[D\to D']}$ is not reduced and hence not almost algebraic.

It is an open problem to find a further condition on the basis weaker than being closed but which is preserved and preserves almost algebraicity under the function space construction.

4.4. Closure under the Plotkin power domain construction

In this section we prove that the c-bifinite domains are closed under the Plotkin power domain construction. Let us first recall the construction of the Plotkin power domain of a continuous cpo.

Definition 44. Let D be a continuous cpo and let B_D be a basis for D. The Plotkin power domain $P_P(D)$ of D is defined as $\mathrm{Idl}(\wp_f^*(B_D), \ll_{EM})$, where $\ll_{EM}$ is the continuous Egli–Milner preorder defined by

$$\begin{aligned} A \ll_{EM} B :\Leftrightarrow\ & \forall b \in B \exists a \in A\ a \ll b \\ & \wedge\ \forall a \in A \exists b \in B\ a \ll b. \end{aligned}$$

The first part of the conjunction is called the Smyth condition and the second part the Hoare condition.

Then $(P_P(D); \subseteq, \perp)$ is acontinuous cpo. It has a basis consisting of principal ideals of the form

$$[A] = \{B \in \wp_f^*(B_D) : B \ll_{EM} A\}, \text{ for } A \in \wp_f^*(B_D).$$

The Plotkin power domain construction is canonical even if there is no canonical choice of base for acontinuous cpo; see [1] for details.

Lemma 45. *Let D be a continuous cpo, let $P_P(D)$ be the Plotkin power domain of D and let $[A]$ and $[B]$ denote elements of a basis B_D for $P_P(D)$. Then the two ordering relations on $P_P(D)$ can be characterised as follows.*

(1) $A \ll_{EM} B \Leftrightarrow [A] \ll [B]$.

(2) *If B_D is almost algebraic then $A \sqsubseteq_{EM} B \Leftrightarrow [A] \sqsubseteq [B]$.*

Proof. We leave the proof to the reader. Note that the implication from left to right in (2) is independent of the assumption that B_D is almost algebraic. □

We now show that the c-bifinite domains are closed under the Plotkin power domain construction.

Theorem 46. *Let D be a c-bifinite domain with a c-bifinite basis B. Then $P_P(D)$ is a c-bifinite domain with c-bifinite basis*

$$B_{P_P(D)} := \{[K] : K \in \wp_f^*(B)\}.$$

Proof. Let $\mathcal{F}$ be a wa-complete cover of B. We show that

$$\mathcal{G} := \{\{[K] : K \in \wp_f^*(A)\} : A \in \mathcal{F}\}$$

is a wa-complete cover of $B_{P_P(D)}$. Let $A \in \mathcal{F}$. We need to show that

$$\{[K] : K \in \wp_f^*(A)\}$$

is wa-complete in $P_P(D)$. Let $C \in \wp_f^*(B)$. It suffices by Lemma 29 to show that there is $B_C \subseteq A$ such that

$$[B_C] = \max(\{[K] : K \in \wp_f^*(A)\} \cap \Downarrow[C]).$$

We define

$$B_C := \{\max(A \cap \Downarrow c) : c \in C\}.$$

We now show that $[B_C] = \max(\{[K] : K \in \wp_f^*(A)\} \cap \Downarrow[C])$. Note that $B_C \subseteq A$, since A is wa-complete. Note also that $B_C \ll_{EM} C$ and hence it follows by Lemma 45(1) that $[B_C] \ll [C]$.

Now suppose that there is $K \in \wp_f^*(A)$ such that $[K] \ll [C]$. It follows by Lemma 45 (i) that $K \ll_{EM} C$. We must show that $[K] \subseteq [B_C]$. It suffices to show that $K \sqsubseteq_{EM} B_C$ by Lemma 45(2). (Note that this direction does not use almost algebraicity.)

We first show that the Hoare condition holds. Let $k \in K$. Then there exists $c_k \in C$ such that $k \ll c_k$. Hence there is $b_k \in B_C$ such that $b_k = \max(A \cap \Downarrow c_k)$. It follows that $k \sqsubseteq b_k$, since A is wa-complete and $K \in \wp_f^*(A)$. Thus the Hoare condition holds.

We now show that the Smyth condition holds. Let $b \in B_C$. Then there exists $c_b \in C$ such that $b = \max(A \cap \Downarrow c_b)$. Since $K \ll_{EM} C$ there is $k_b \in K$ such that $k_b \ll c_b$. Note that $k_b \in A$. Then $k_b \sqsubseteq b$ by the definition of B_C. It follows that $K \sqsubseteq_{EM} B_C$ and hence $[B_C] = \max(\{[K] : K \in \wp_f^*(A)\} \cap \Downarrow[C])$. Thus

$$\{[K] : K \in \wp_f^*(A)\}$$

is wa-complete in $P_P(D)$.

For each finite sequence $B_0, B_1, \ldots, B_k \subseteq_f B$ there is some $A \in \mathcal{F}$ such that $\bigcup_{i \leqslant n} B_i \subseteq A$, since D is a c-bifinite domain. This shows that $\mathcal{G}$ is a wa-complete cover of $B_{P_P(D)}$ and hence that $P_P(D)$ is a c-bifinite domain with c-bifinite basis $B_{P_P(D)}$. □

We now show that the Plotkin power domain of a continuous cpo D has an almost algebraic basis if D has an almost algebraic basis.

Theorem 47. *Let D be a continuous cpo with basis B_D and suppose that B_D is almost algebraic. Then the Plotkin power domain $P_P(D)$ has an almost algebraic basis $B_{P_P(D)} := \{[K] : K \in \wp_f^*(B_D)\}$.*

Proof. We show that $B_{P_P(D)}$ is almost algebraic. To prove the first condition, let $A \subseteq_f^* B_D$. For each $a \in A$ we choose an almost algebraic sequence $(a^j)_j$ for a. Define for $j \in \omega$ the set $A^j = \{a^j : a \in A\}$. We claim that $([A^j])_j$ is an almost algebraic sequence for $[A]$. Clearly, $[A] \ll [A^{j+1}] \ll [A^j]$ for each $j \in \omega$. Let $[C] \gg [A]$, i.e. $A \ll_{EM} C$, where $C \subseteq_f^* B_D$. For each $c \in C$ there is $a_c \in A$ such that $a_c \ll c$. Hence there is j_c such that

$$j \geqslant j_c \Rightarrow a_c \ll a_c^j \ll c.$$

On the other hand, for each $a \in A$ there is $c_a \in C$ such that $a \ll c_a$, and hence there is j_a such that

$$j \geqslant j_a \Rightarrow a \ll a^j \ll c_a.$$

Let

$$j_0 \geqslant \max(\{j_c : c \in C\} \cup \{j_a : a \in A\}).$$

Then clearly $A^{j_0} \ll_{EM} C$, i.e. $[A^{j_0}] \ll [C]$. This shows that $B_{P_P(D)}$ is almost algebraic (1).

To prove the second condition, let $A, B \subseteq^*_f B_D$ be such that $\Uparrow[A] \subseteq \Uparrow[B]$. We need to show $[B] \subseteq [A]$. By Lemma 45 it is sufficient to show that $B \sqsubseteq_{EM} A$. We first show the Smyth condition. Let $A^j \gg_{EM} A$ be the sequences obtained above from A. Hence we have $A^j \gg_{EM} B$. Fix $a \in A$. Thus for all j there is an element $b_j \in B$ such that $a^j \gg b_j$. Since B is finite there exists $b \in B$ such that $a^j \gg b$ holds for all j. Hence we have $\Uparrow a \subseteq \Uparrow b$ and it follows that $b \sqsubseteq a$, since B_D is almost algebraic (2). This shows the Smyth condition.

We now show the Hoare condition. Fix $b \in B$. Then for all j there is $a^j \in A^j$ such that $a^j \gg b$. Since A is finite there is some $a \in A$ such that $a^j \gg b$ holds for all j. Then we have $\Uparrow a \subseteq \Uparrow b$ and hence $b \sqsubseteq a$ by the assumption that B_D is almost algebraic (2). This shows the Hoare condition and thus it follows that $B_{P_P(D)}$ is almost algebraic (2). □

We note the following corollary.

Corollary 48. *The class of continuous cpos with an almost algebraic and c-bifinite basis is closed under the Plotkin power domain construction.*

Proof. By Theorems 46 and 47. □

5. Effective c-bifinite domains

In this section we develop an effectivity notion for c-bifinite domains.

5.1. Basic definitions

In this section we present the definition of an effective c-bifinite domain. It is a generalisation of the definition of an effective bifinite domain in [11].

Definition 49. Let $(D; \sqsubseteq, \bot)$ be acontinuous cpo and let B be a c-bifinite basis for D. We say that B is effective if there is a numbering $\alpha : \omega \to B$ such that
(1) $\sqsubseteq$ on B is α-decidable;
(2) $\ll$ on B is α-decidable; and
(3) there is an $\tilde{\alpha}$-decidable wa-complete cover $\mathcal{F}$ of B.
We call (D, α) an effective c-bifinite domain if D has an effective c-bifinite basis.

In order to prove an effective version of Theorem 42 we need two lemmas. First we prove a lemma that gives a characterisation of the wa-complete subsets of a wa-complete set.

Lemma 50. *Let D be a continuous cpo and let B_D be an almost algebraic basis for D. Suppose that $A \subseteq_f B_D$ is wa-complete. Then $B \subseteq A$ is wa-complete if and only if*

$$(\forall C \subseteq B)(\forall a \in A)(\exists b \in B)(C \sqsubseteq a \Rightarrow C \sqsubseteq b \sqsubseteq a).$$

Proof. Let $A \subseteq_f B_D$ be wa-complete. For the if direction we let $B \subseteq A$ and assume that the condition holds for B. Let $x \in D$, let $C = \{b \in B : b \ll x\}$ and let $a := \max(A \cap \Downarrow x)$. Then $C \sqsubseteq a$ and hence the condition guarantees the existence of a $b \in B$ such that $C \sqsubseteq b \sqsubseteq a$. Clearly, $b = \max(B \cap \Downarrow x)$ and hence B is wa-complete.

For the only if direction we need to use that B_D is almost algebraic. Let $C \subseteq B$, $a \in A$, $C \sqsubseteq a$ and let $(a_n)_n$ be an almost algebraic sequence for a. Suppose that B is wa-complete. Then there exists $b_n \in B$ such that $C \sqsubseteq b_n \ll a_n$, for each $n \in \omega$. But since B is finite there exists $b \in B$ such that $C \sqsubseteq b \ll a_n$ holds, for each $n \in \omega$. It follows that $b \sqsubseteq a$, since B_D is assumed to be almost algebraic. $\square$

We note the following corollary.

Corollary 51. *Let $D = (D; \sqsubseteq, \perp)$ be a continuous cpo and let $\alpha : \omega \to B_D$ be a numbering such that $\sqsubseteq$ on B_D is α-decidable. Suppose that B_D is almost algebraic and that $\mathcal{F}$ is an $\tilde{\alpha}$-decidable wa-complete cover of B_D. Then there is an $\tilde{\alpha}$-decidable wa-complete cover of B_D, defined by*

$$\mathcal{G} := \{B \in \wp_f(B_D) : B \text{ wa-complete}\}.$$

Proof. Given $B \in \wp_f(B_D)$ we search for $A \in \mathcal{F}$ such that $B \subseteq A$. This is effective since $\sqsubseteq$ is α-decidable. We then use the condition in Lemma 50 to decide whether B is wa-complete. $\square$

The following lemma says that we can extend the domain of a base function to a larger wa-complete set, without changing the value of the function.

Lemma 52. *Let D and E be continuous cpos with bases B_D and B_E, respectively, and suppose that B_D is almost algebraic. Let v be c-joinable and let $A \subseteq_f B_D$ be a wa-complete set such that $\pi_0(v) \subseteq A$. Then there is a c-joinable set u such that $\pi_0(u) = A$ and $s_u = s_v$.*

Proof. Let $v = \{(b, d_b) : b \in B\}$, where $B = \pi_0(v)$. Fix $a \in A$ and let $(a^n)_n$ be an almost algebraic sequence for a. Define $b_a := \max(B \cap \mathord{\downarrow\!\!\downarrow} a^{\bar{n}})$, where $\bar{n}$ is such that if $n \geqslant \bar{n}$ then

$$\max(B \cap \mathord{\downarrow\!\!\downarrow} a^{\bar{n}}) = \max(B \cap \mathord{\downarrow\!\!\downarrow} a^n).$$

We define $u := \{(a, d_{b_a}) : a \in A\}$ and claim that u is c-joinable. Suppose that $a \sqsubseteq a' \in A$. Then $b_a \sqsubseteq b_{a'}$ holds and hence it follows that $d_{b_a} \sqsubseteq d_{b_{a'}}$. This proves the claim, since A is assumed to be wa-complete.

We now show that $s_u = s_v$. Let $x \in D$ and let $a_x = \max(A \cap \mathord{\downarrow\!\!\downarrow} x)$ and $b_x = \max(B \cap \mathord{\downarrow\!\!\downarrow} x)$. It suffices to show that $b_{a_x} = b_x$. For all large n we have $a_x \ll a_x^n \ll x$ and hence $b_{a_x} \sqsubseteq b_x$. On the other hand, $b_x \sqsubseteq a_x \ll a_x^n$ for each n, since $B \subseteq A$. But then $b_x \sqsubseteq b_{a_x}$, i.e. $b_x = b_{a_x}$.

As a result we obtain a basis for $[D \to E]$ with basis elements formed from a wa-complete cover of B_D. $\square$

Corollary 53. *Let D and E be c-bifinite domains with c-bifinite bases B_D and B_E, respectively, and suppose that B_D is almost algebraic and that $\mathcal{F}$ is a wa-complete cover of B_D. Then*

$$\{s_u : u \in \wp_f^*(B_D \times B_E) \wedge \pi_0(u) \in \mathcal{F} \wedge u \text{ c-joinable}\}$$

is a basis for $[D \to E]$. Furthermore, if B_E is countable then the basis is c-bifinite.

Proof. This follows immediately from Lemmas 52, 35 and Theorem 42. $\square$

5.2. Two theorems for effective c-bifinite domains

In this section we prove two theorems for effective c-bifinite domains on the closure under the function space and the Plotkin power domain construction. First, we present the generalisation of Theorem 42.

Theorem 54. *Let (D, α) and (E, β) be effective c-bifinite domains with effective c-bifinite bases B_D and B_E, respectively. Suppose that B_D is almost algebraic. Then there is a numbering γ of a basis $B_{[D \to E]}$ of $[D \to E]$* such that *$([D \to E], \gamma)$ is an effective c-bifinite domain. Furthermore, f is a computable element of the function space if and only if f is an effective function.*

Proof. Let $\mathcal{F} = \{A_i : i \in \omega\}$ and $\mathcal{G} = \{B_i : i \in \omega\}$ be the $\tilde{\alpha}$-decidable and $\tilde{\beta}$-decidable wa-complete covers of B_D and B_E, respectively. By Corollary 53 above we have that

$$B_{[D \to E]} = \{s_u : u \in \wp_f^*(B_D \times B_E) \wedge \pi_0(u) \in \mathcal{F} \wedge u \text{ c-joinable}\}$$

is a basis for $[D \to E]$. We construct a numbering γ of $B_{[D\to E]}$ in an informal manner. Denote $\widetilde{\alpha \times \beta}(j)$ by $\{(a_l, b_l) : l \in K_j\}$. We define the recursive domain Ω_γ of γ by

$$\langle i, j\rangle \in \Omega_\gamma \Leftrightarrow \{(a_l, b_l) : l \in K_j\} \text{ satisfies the monotonicity requirement}$$
$$\text{in the definition of c-joinable and } \{a_l : l \in K_j\} = A_i.$$

Define $\gamma : \Omega_\gamma \to B_{[D\to E]}$ by $\gamma(\langle i, j\rangle) = s_u$, where $u = \{(a_l, b_l) : l \in K_j\}$. Extend the domain of γ to ω by $\gamma(\langle i, j\rangle) = \perp_{[D\to E]}$ if $\langle i, j\rangle \notin \Omega_\gamma$. Note that we have computable access via the coding γ to all the ingredients of the elements in the basis. We can thus take the liberty to reason informally from here.

We obtain that $\ll$ on $B_{[D\to E]}$ is γ-decidable by Lemma 41 and the equivalence

$$s_u \ll s_v \Leftrightarrow (\forall (a, b) \in u)(\langle a; b\rangle \ll s_v)$$
$$\Leftrightarrow (\forall (a, b) \in u)(b \ll s_v(a)),$$

where the last expression is decidable since we have effective c-bifinite domains (D, α) and (E, β).

To show that $\sqsubseteq$ on $B_{[D\to E]}$ is γ-decidable we first note that

$$s_u \sqsubseteq s_v \;\Leftrightarrow\; (\forall (a, b) \in u)(\langle a; b\rangle \sqsubseteq s_v).$$

Next we claim that

$$\langle a; b\rangle \sqsubseteq s_v \;\Leftrightarrow\; (b = \perp_E \vee (\exists (c, d) \in v)(c \sqsubseteq a \wedge b \sqsubseteq d)).\ (*)$$

Note that the decidability of $\sqsubseteq$ follows from the claim.

The direction right to left is immediate and hence we concentrate on the direction left to right. Assume that $b \neq \perp_E$. Let $(a, b) \in u$ and choose an almost algebraic decreasing sequence $(a_n)_n \gg a$ for a, which exists since B_D is almost algebraic. For each $x \in D$ we note that if $a \ll x$ then we can find an index i such that $a_i \ll x$. Hence it follows that $C^i := \{c \in \pi_0(v) : c \ll a_i\} \neq \emptyset$. Since $\pi_0(v)$ is finite there must be an index $n_0 \in \omega$ such that $C^n = C^{n+1}$ holds, for each $n \geqslant n_0$. Let $c = \max(\pi_0(v) \cap \mathop{\downarrow\!\!\downarrow} a_{n_0})$. Since B_D is almost algebraic (2) we know that $c \sqsubseteq a$. Furthermore, $\langle a; b\rangle \sqsubseteq s_v$ and hence

$$\langle a; b\rangle(a_{n_0}) = b \sqsubseteq s_v(a_{n_0}) = \langle c; d\rangle(a_{n_0}) = d,$$

where $(c, d) \in v$. This shows that the second part of the disjunction holds and thus we have proved the equivalence $(*)$.

The next step is to construct a $\tilde{\gamma}$-decidable cover of $B_{[D\to E]}$. Given $A \in \mathcal{F}$ and $B \in \mathcal{G}$ we define

$$M_{AB} = \{s_v : v \subseteq A \times B \wedge v \text{ c-joinable} \wedge \pi_0(v) = A\}.$$

Clearly we have that the set $\mathcal{H} = \{M_{AB} : A \in \mathcal{F}, B \in \mathcal{G}\}$ is $\tilde{\gamma}$-decidable. By the argument of Theorem 42 we see that each M_{AB} is wa-complete and hence that $\mathcal{H}$ is a wa-complete cover. Thus $([D \to E], \gamma)$ is an effective c-bifinite domain.

We finish the proof by observing that for each c-joinable set $u \subseteq_f B_D \times B_E$ and for each function $f \in [D \to E]$ we have

$$s_u \ll f \Leftrightarrow (\forall (a, b) \in u)(\langle a; b\rangle \ll f)$$
$$\Leftrightarrow (\forall (a, b) \in u)(b \ll f(a)).$$

It follows that f is (α, β)-effective if and only if f is γ-computable. □

We now show an effective version of Theorem 46.

Theorem 55. *Let (D, α) be an effective c-bifinite domain with an effective c-bifinite and almost algebraic basis B_D. Then $(P_P(D), \tilde{\alpha})$ is an effective c-bifinite domain with an effective c-bifinite and almost algebraic basis*

$$B_{P_P(D)} := \{[K] : K \in \wp_f^*(B_D)\}.$$

Proof. We examine the proof of Theorem 46 and note that the cover $\mathcal{G}$ is $\tilde{\tilde{\alpha}}$-decidable. We also note that $\ll$ and $\sqsubseteq$ on $B_{P_P(D)}$ are $\tilde{\alpha}$-decidable by Lemma 45. □

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